MATH 2230 A

Tutorial 6 19-10-2016

Topics: ◦ Revision exercise for Mid-term.

1) Find $\cos 4\theta$ and $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Ans: By De Moivre's formula,

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= e^{i4\theta} \\
&= (e^{i\theta})^4 \\
&= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta \\
&\quad - 4i\cos\theta\sin^3\theta + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\
&\quad + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)
\end{aligned}$$

By comparing the real and imaginary parts on both sides, we have

$$\begin{cases} \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ \sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \end{cases}$$

2) Show that for $a, b \in \mathbb{R}$,

a) $$\sum_{k=0}^{n-1} \sin(a+kb) = \frac{\sin\frac{nb}{2}}{\sin\frac{b}{2}} \sin\left(a + \frac{(n-1)b}{2}\right)$$

b) $$\sum_{k=0}^{n-1} \binom{n}{k} \sin(a+kb) = 2^n \left(\cos\frac{b}{2}\right)^n \sin\left(a + \frac{nb}{2}\right)$$

Ans: a) Note that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \cos(a+kb) + i\sum_{k=0}^{n-1} \sin(a+kb) \\
&= \sum_{k=0}^{n-1} \left(\cos(a+kb) + i\sin(a+kb)\right) \\
&= \sum_{k=0}^{n-1} e^{i(a+kb)} \\
&= e^{ia} \sum_{k=0}^{n-1} e^{kbi}
\end{aligned}$$

Hence we have

$$\sum_{k=0}^{n-1} \cos(a+kb) + i\sum_{k=0}^{n-1} \sin(a+kb) = e^{ia}\sum_{k=0}^{n-1} e^{kbi}$$

$$= e^{ia}\frac{1-(e^{bi})^n}{1-e^{bi}}$$

$$= e^{ia}\frac{1-e^{nbi}}{1-e^{bi}}$$

$$= e^{ia}\frac{e^{\frac{nbi}{2}}-e^{-\frac{nbi}{2}}}{e^{\frac{bi}{2}}-e^{-\frac{bi}{2}}} \cdot \frac{e^{\frac{nbi}{2}}}{e^{\frac{bi}{2}}}$$

$$= e^{i(a+\frac{(n-1)b}{2})}\frac{\sin\frac{nb}{2}}{\sin\frac{b}{2}}$$

By comparing the imaginary parts on both sides, we have

$$\sum_{k=0}^{n-1} \sin(a+kb) = \frac{\sin\frac{nb}{2}}{\sin\frac{b}{2}}\sin\left(a+\frac{(n-1)b}{2}\right)$$

b) Recall that $(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k$.

So $\sum_{k=0}^{n} \binom{n}{k}\cos(a+kb) + i\left(\sum_{k=0}^{n}\binom{n}{k}\sin(a+kb)\right)$

$$= \sum_{k=0}^{n}\binom{n}{k} e^{i(a+kb)}$$

$$= e^{ia}\sum_{k=0}^{n}\binom{n}{k}(e^{bi})^k$$

$$= e^{ia}(1+e^{bi})^n$$

$$= e^{ia}\left(e^{\frac{bi}{2}}(e^{\frac{bi}{2}}+e^{-\frac{bi}{2}})\right)^n$$

$$= e^{ia}e^{\frac{nbi}{2}}\left(2\cos\left(\frac{b}{2}\right)\right)^n$$

$$= 2^n\left(\cos\frac{b}{2}\right)^n e^{i(a+\frac{nb}{2})}$$

By comparing the imaginary parts on both sides, we have

$$\sum_{k=0}^{n}\binom{n}{k}\sin(a+kb) = 2^n\left(\cos\frac{b}{2}\right)^n\sin\left(a+\frac{nb}{2}\right)$$

3) Let $f(z)=\bar{z}e^{-|z|^2}$

Find the points at which f is complex differentiable and compute its derivative at these points.

Ans: $f(z)=\overline{(x+iy)}e^{-(x^2+y^2)} = xe^{-(x^2+y^2)} - iye^{-(x^2+y^2)}$

Let $u(x,y)=xe^{-(x^2+y^2)}$ and $v(x,y)=-y\cdot e^{-(x^2+y^2)}$

Then we have

$$\begin{cases} u_x = e^{-(x^2+y^2)} + xe^{-(x^2+y^2)}(-2x) = (1-2x^2)e^{-(x^2+y^2)} \\ u_y = -2xye^{-(x^2+y^2)} \\ v_x = 2xye^{-(x^2+y^2)} \\ v_y = -e^{-(x^2+y^2)} - ye^{-(x^2+y^2)}(-2y) = -(1-2y^2)e^{-(x^2+y^2)} \end{cases}$$

The CR-eqtn holds iff

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} 1-2x^2 = -1+2y^2 \\ 2xye^{-(x^2+y^2)} = 2xye^{-(x^2+y^2)} \end{cases}$$

$$\Rightarrow x^2+y^2 = 1$$

Since the partial derivatives are continuous and the CR-eqtn holds on $\{z \mid |z|=1\}$, f is complex diff. there and

$$f'(z) = u_x + iv_x = (1-2x^2)e^{-(x^2+y^2)} + 2xyie^{-(x^2+y^2)}$$

Since $|z|=1$, $f'(z) = e^{-1}(1-2x^2) + 2e^{-1}xyi$.

4) Solve the following equations:

a) $z^i = 1$ in principal branch;

b) $\sinh(\bar{z}) = 1$

Ans: a) Note that $z^i = e^{i\,\mathrm{Log}\,z}$

$$= e^{i(\ln|z| + i\,\mathrm{Arg}(z))}$$

$$= e^{-\mathrm{Arg}(z)} \cdot e^{i\ln|z|}$$

So $z^i = 1$

$$\Leftrightarrow \begin{cases} -\mathrm{Arg}(z) = 0 \\ \ln|z| = 2n\pi \end{cases}, n \in \mathbb{Z}$$

$$\Leftrightarrow \quad z = e^{2n\pi}, n \in \mathbb{Z}.$$

b)

$$\sinh(\bar{z}) = 1$$

$$\frac{e^{\bar{z}} - e^{-\bar{z}}}{2} = 1$$

$$e^{\bar{z}} - e^{-\bar{z}} = 2$$

$$e^{2\bar{z}} - 1 = 2e^{\bar{z}}$$

$$e^{2\bar{z}} - 2e^{\bar{z}} - 1 = 0$$

$$e^{\bar{z}} = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= 1 \pm \sqrt{2}$$

$\therefore \bar{z} = \log(1+\sqrt{2})$ or $\bar{z} = \log(1-\sqrt{2})$

$\bar{z} = \ln(1+\sqrt{2}) + 2n\pi i$ or $\bar{z} = \ln|1-\sqrt{2}| + (2n\pi - \pi)i$

$z = \ln(1+\sqrt{2}) - 2n\pi i$ or $z = \ln|1-\sqrt{2}| - (2n-1)\pi i$,

where $n \in \mathbb{Z}$.

5) Show that a non-zero entire function f must be a constant if

a) $\mathrm{Arg}\, f = \text{constant}$; or

b) $uv = \text{constant}$; or

c) $au + bv + c = 0$, where $a, b, c \in \mathbb{R}$ and a, b are not simultaneously zero.

Ans: a) Let $f(z) = r(z)e^{i\theta}$, where $\theta = \operatorname{Arg} f$ = constant.
$$= r(z)\cos\theta + i\, r(z)\sin\theta.$$
Let $u(x,y) = r(x,y)\cos\theta$ and $v(x,y) = r(x,y)\sin\theta$.
By CR-eqtn, we have
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} r_x\cos\theta = r_y\sin\theta \\ r_y\cos\theta = -r_x\sin\theta \end{cases}$$
$$\Rightarrow \begin{cases} \cos\theta\, r_x - \sin\theta\, r_y = 0 \\ \sin\theta\, r_x + \cos\theta\, r_y = 0 \end{cases}$$
$$\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
As $\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = 1 \neq 0$, we have $r_x = 0 = r_y$.

Hence r and $f = re^{i\theta}$ are constants.

b) Let $f(z) = u + iv$.
By CR-eqtn, we have $u_x = v_y$ and $u_y = -v_x$.
As uv = constant, we have
$$u_x v + u v_x = 0 \quad \text{—— (1)}.$$
Also we have $u_y v + u v_y = 0$.
By CR-eqtn, we have $-v_x v + u u_x = 0$ —(2).
From (1) and (2), we have
$$\begin{pmatrix} v & u \\ u & -v \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
As $\det\begin{pmatrix} v & u \\ u & -v \end{pmatrix} = -(u^2+v^2) \neq 0$, we have $u_x = 0 = v_x$

By CR-eqtn we also have $u_y = 0 = v_y$.
Hence f is constant.

c) By assumption, we have
$$au + bv + c = 0.$$
$$\Rightarrow \begin{cases} a u_x + b v_x = 0 & \text{—— (1)} \\ a u_y + b v_y = 0 & \text{—— (2)} \end{cases}$$

Also, by CR-eqtn we have

$$\begin{cases} u_x - v_y = 0 & \text{---}(3) \\ u_y + v_x = 0 & \text{---}(4). \end{cases}$$

From (1) - (4), we have

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By direct calculation, one can find that

$$\det \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = a^2 + b^2 > 0.$$

Hence we have $u_x = u_y = v_x = v_y = 0$.
So f is constant.

Remark: In 5)b), for the system of linear equations

$$\begin{pmatrix} v & u \\ u & -v \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we also need to consider the case where

$$\det \begin{pmatrix} v & u \\ u & -v \end{pmatrix} = -(u^2 + v^2) = 0.$$

In such case, we have $u = v = 0$ and $f = u + iv = 0$.
Since it is given that f is non-zero, contradiction arise. So $\det \begin{pmatrix} v & u \\ u & -v \end{pmatrix} = -(u^2 + v^2) \neq 0$.

In mid-term and final exam, you have to consider every cases. Otherwise some marks will be deducted.